

# Generalized fractional operators for nonstandard Lagrangians\*

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## Abstract

In this note we study the application of generalized fractional operators to a particular class of nonstandard Lagrangians. These are typical of dissipative systems and the corresponding Euler–Lagrange and Hamilton equations are analyzed. The dependence of the equation of motion on the generalized kernel permits to obtain a wide range of different configurations of motion. Some examples are discussed and analyzed.

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## 1 Introduction

Fractional calculus plays an important role in the study of different problems in physics, engineering, finance, and many other branches of science [1]. Its formulation dates back to the 19th century, but its applications are, surprisingly, very recent. One of the most successful results of fractional calculus concerns the description of anomalous diffusion [2] and a promising research topic is the fractional variational calculus [3]. Close to the fractional variational principle, is

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the fractional action-like variational approach (FALVA), where the Lagrangian (eventually containing fractional derivative terms) is weighted by a power law function [4, 5]. Recent works posed the attention on the application of a generalized kernel in the action [6–10]. For a survey see [11]. In this note we follow this approach, applied to nonstandard Lagrangians. Such kind of Lagrangians cannot be described by a simple difference between kinetic and potential energies, and are typical to dissipative systems [12]. The text is organized as follows: in Section 2 we review the necessary notions and results from generalized fractional calculus. Our results appear in Section 3, where we obtain and discuss the generalized Euler–Lagrange equations to nonstandard Lagrangians, considering different kernels. Our results extend those of [13, 14]. We end with Section 4 of conclusion.

## 2 Preliminaries

In this section we briefly review the main notions regarding generalized fractional operators. For details and for proofs on the generalized fractional calculus of variations we refer the reader to [6, 7, 10]. Let us consider a function  $l(\tau)$  and another function  $k_\alpha(t, \tau)$ , called the kernel, eventually depending on  $\alpha$ . Throughout the text we assume, if not differently specified, that

- $0 < \alpha < 1$ ;
- $t \in [a, b]$ ;
- $\tau \in (a, t)$ .

Following [7, 15], we make use of the following definition.

**Definition 2.1.** *The generalized fractional operator  $S_P^\alpha$  is given by*

$$S_P^\alpha[l](t) = p \int_a^t k_\alpha(t, \tau) l(\tau) d\tau + q \int_t^b k_\alpha(\tau, t) l(\tau) d\tau, \quad (1)$$

where  $p$  and  $q$  are two real numbers,  $P = \langle a, t, b, p, q \rangle$ , and  $k_\alpha(t, \tau)$  is the kernel.

It is worth noting that if  $P = \langle a, t, b, 1, 0 \rangle$  and

$$k_\alpha(t, \tau) = \frac{1}{\Gamma(\alpha)} (t - \tau)^{\alpha-1},$$

where  $\Gamma$  is the Gamma function, then (1) reduces to

$$S_P^\alpha[l](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} l(\tau) d\tau.$$

In case  $l$  is a Lagrangian, this kind of operator is used to derive the fractional Euler–Lagrange equations, constituting the so-called fractional action-like variational approach (FALVA) [4, 5]. In this case,  $S_P^\alpha$  is the left Riemann–Liouville

fractional integral  ${}_a I_t^\alpha$  [16]:

$${}_a I_t^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \quad (2)$$

The following theorem is essential to obtain the Euler–Lagrange equations in case of generalized kernels.

**Theorem 2.1** (Theorem 3.1 of [7]). *Let us consider  $k_\alpha$  to be a square-integrable function in  $\Delta = [a, b] \times [a, b]$ ,  $l, m \in L_2([a, b])$ , and  $P = \langle a, t, b, p, q \rangle$ . Then  $S_P^\alpha$  satisfies the following integration by parts formula:*

$$\int_a^b m(t) S_P^\alpha[l](t) dt = \int_a^b l(t) S_{\hat{P}}^\alpha[m](t) dt, \quad (3)$$

where  $\hat{P} = \langle a, t, b, q, p \rangle$ .

If  $k_\alpha$  satisfies the property  $k_\alpha(t, \tau) = k_\alpha(t - \tau)$ , as is the case for the Riemann–Liouville fractional integral  ${}_a I_t^\alpha$  (2), then the integration by parts formula (3) holds for two functions  $l(t)$  and  $m(t)$  under hypotheses as stated in the following theorem.

**Theorem 2.2** (Theorem 3.2 of [7]). *If  $k_\alpha(t, \tau) = k_\alpha(t - \tau)$ ,  $l \in L_1([a, b])$  and  $m \in C([a, b])$ , then operator  $S_P^\alpha$  satisfies the integration by parts formula (3).*

### 3 Main Results

We investigate Euler–Lagrange equations for actions involving generalized kernels.

**Definition 3.1.** *The generalized fractional action  $\mathcal{A}(x)$  is given by*

$$\mathcal{A}(x) = S_{P_1}^\alpha[L](b) = \int_a^b k_\alpha(b, \tau) L(\tau, x(\tau), \dot{x}(\tau)) d\tau \quad (4)$$

with boundary conditions

$$x(a) = x_a, \quad x(b) = x_b, \quad (5)$$

where  $P_1 = \langle a, b, b, 1, 0 \rangle$  and  $L(\tau, x(\tau), \dot{x}(\tau))$  is the Lagrangian.

We consider the problem of finding a function  $x$  that minimizes the functional  $\mathcal{A}(\cdot)$  subject to boundary conditions (5). As a corollary of [7, Theorem 4.2], we obtain the following result.

**Theorem 3.1** (Generalized fractional Euler–Lagrange equations associated with (4)). *Let  $x$  be a solution to the problem of finding a function  $x$  that minimizes the functional  $\mathcal{A}$  subject to boundary conditions (5). If  $k_\alpha(b, \tau)$  satisfies the conditions of Theorem 2.1 or Theorem 2.2, together with*

- $L \in C^1([a, b] \times \mathbb{R}^2; \mathbb{R})$ ,
- $k_\alpha(b, \tau), \partial_3 L \in AC([a, b])$ ,
- $k_\alpha(b, \tau), \partial_2 L \in C([a, b])$ ,

where  $\partial_i$  is the partial derivative with respect to the  $i$ th argument of  $L$ , then the following generalized fractional Euler–Lagrange equations hold:

$$\partial_2 L(\tau, x(\tau), \dot{x}(\tau)) - \frac{d}{d\tau} \partial_3 L(\tau, x(\tau), \dot{x}(\tau)) = \frac{dk_\alpha(b, \tau)}{d\tau} \frac{\partial_3 L(\tau, x(\tau), \dot{x}(\tau))}{k_\alpha(b, \tau)} \quad (6)$$

for all  $\tau \in [a, b]$ .

### 3.1 Nonstandard Lagrangians

When a Lagrangian is expressed as the difference between kinetic and potential energy, it is called a *standard Lagrangian*. If it is not possible to discriminate the two contributions of energy, then the Lagrangian is said to be a *nonstandard Lagrangian* [17]. An interesting purpose is to find equations of motion able to describe dissipative dynamical systems by a nonstandard Lagrangian. An equation of motion of form

$$\ddot{x}(\tau) + A(\tau)\dot{x}(\tau) + B(\tau)x(\tau) = 0, \quad (7)$$

where  $\dot{x}(\tau) = \frac{dx(\tau)}{d\tau}$  and  $A(\tau)$  and  $B(\tau)$  are arbitrary, but continuous, differentiable and integrable functions, is typical of unforced dissipative systems. As shown by [14], equation (7) can be derived from a nonstandard Lagrangian with time-dependent coefficients:

$$L(\tau, x(\tau), \dot{x}(\tau)) = \frac{1}{r(\tau)\dot{x}(\tau) + s(\tau)x(\tau)}, \quad (8)$$

where  $r(\tau)$  and  $s(\tau)$  are continuous and at least twice differentiable functions. The coefficients  $A(\tau)$  and  $B(\tau)$  are related to  $r(\tau)$  and  $s(\tau)$  by the solution of a nonlinear second-order Riccati equation [14]. We next obtain, and discuss, the generalized fractional Euler–Lagrange equations for the nonstandard Lagrangian with time-dependent coefficients (8).

Let us take the Lagrangian (8) and insert it in (6). We obtain the equation of motion

$$\ddot{x} + \frac{\dot{x}}{2} \left[ \frac{3s}{r} + \frac{\dot{r}}{r} - \frac{\dot{k}_\alpha}{k_\alpha} \right] + x \left[ \frac{s^2}{2r^2} - \frac{\dot{r}s}{2r^2} + \frac{\dot{s}}{r} - \frac{s}{2r} \frac{\dot{k}_\alpha}{k_\alpha} \right] = 0, \quad (9)$$

where  $\dot{r}(\tau) = \frac{dr(\tau)}{d\tau}$ ,  $\dot{s}(\tau) = \frac{ds(\tau)}{d\tau}$  and  $\dot{k}_\alpha(b, \tau) = \frac{dk_\alpha(b, \tau)}{d\tau}$ . This equation is different from the classical: the fractional version consists in the presence of the term  $-\frac{\dot{x}}{2} \left[ \frac{\dot{k}_\alpha}{k_\alpha} \right] - x \left[ \frac{s}{2r} \frac{\dot{k}_\alpha}{k_\alpha} \right]$  [14]. In contrast, our equation of motion (9) consists

of a friction term and a harmonic term, both time depending. We note that in case  $r$  and  $s$  are constant in time, the equation of motion (9) reduces to

$$\ddot{x} + \frac{\dot{x}}{2} \left[ \frac{3s}{r} - \frac{\dot{k}_\alpha}{k_\alpha} \right] + x \left[ \frac{s^2}{2r^2} - \frac{s}{2r} \frac{\dot{k}_\alpha}{k_\alpha} \right] = 0. \quad (10)$$

It is worth noting that if  $\frac{\dot{k}_\alpha}{k_\alpha} = \frac{3s}{r}$ , then we get the equation of an undamped oscillator. Nevertheless, this leads to a nonphysical solution, corresponding to a negative coefficient multiplying  $x$ . Thus, at least for constant  $r$  and  $s$ , this kind of Lagrangian describes exclusively dissipative systems. To get the equation of a damped harmonic oscillator, that is, with coefficients multiplying  $x$  and  $\dot{x}$  both positive, it is convenient to consider the sign of the ratio  $\frac{\dot{k}_\alpha}{k_\alpha}$ . It is easy to show that if  $\frac{\dot{k}_\alpha}{k_\alpha} > 0$ , then the following condition holds:

$$0 < \frac{\dot{k}_\alpha}{k_\alpha} < \frac{3s}{r}$$

for all  $\tau \in [a, b]$ . This poses serious limitations in the behavior of the kernel. In case  $\frac{\dot{k}_\alpha}{k_\alpha} < 0$ , physical solutions are obtained only if  $\frac{s}{r} > 0$  for all  $\tau \in [a, b]$ . The case  $\frac{\dot{k}_\alpha}{k_\alpha} = 0$  refers to the classical Euler–Lagrange equations.

### 3.2 Hamilton formalism

Let us consider the Hamilton formalism for a nonstandard Lagrangian  $L$  as in (8), with  $r$  and  $s$  constant in time and  $\frac{s}{r} > 0$ . The Hamiltonian is defined by

$$H(\tau, x(\tau), p(\tau)) = p(\tau)\dot{x}(\tau) - L(\tau, x(\tau), \dot{x}(\tau)),$$

where  $p(\tau) = \partial_3 L(\tau, x(\tau), \dot{x}(\tau))$ . Being

$$dH = \partial_1 H \, d\tau + \partial_2 H \, dx + \partial_3 H \, dp,$$

where  $\partial_i$  is the partial derivative with respect to the  $i$ th argument of  $H$ , we have

$$dH = -\partial_1 L \, d\tau - \partial_2 L \, dx + \dot{x} \, dp$$

and

$$\partial_1 H = \partial_1 L, \quad \partial_2 H = -\partial_2 L, \quad \partial_3 H = \dot{x}. \quad (11)$$

At this point, the Hamilton equations follow. The momentum  $p$  is

$$p = \partial_3 L = -\frac{r}{(r\dot{x} + sx)^2} \quad (12)$$

and the Hamiltonian  $H$

$$H = -\frac{sxp}{r}. \quad (13)$$

Thus, (11) can be written as

$$\partial_3 H = -\frac{sx}{r}, \quad \partial_2 H = -\frac{sp}{r}, \quad \partial_1 H = -\frac{s(\dot{x}p + x\dot{p})}{r}.$$

Making use of (6) and (11), the Hamilton equations for  $\dot{p} = \partial p / \partial \tau$  and  $\dot{x}$  are

$$\begin{aligned} \dot{p} &= -\partial_2 H - \frac{\dot{k}_\alpha}{k_\alpha} p = \frac{sp}{r} - \frac{\dot{k}_\alpha}{k_\alpha} p, \\ \dot{x} &= \partial_3 H = -\frac{sx}{r}. \end{aligned} \tag{14}$$

It is interesting to note that the Hamiltonian (13) is written in a much simpler form than the Lagrangian (8) and if  $r = s > 0$ , for all  $\tau \in [a, b]$ , then the sign of  $H$  depends exclusively on  $x$  ( $H > 0$  if  $x > 0$ ,  $H < 0$  otherwise). Moreover, it is easy to show that differentiation  $p$  (12) in time and then developing it in the third member of (14), as in (12), it is possible to obtain (10), that is, the equivalence between Euler–Lagrange and Hamilton equations is proven.

## 4 Conclusion

In this brief note we discussed the application of the recent generalized fractional calculus of variations [6, 7] to Euler–Lagrange equations of nonstandard Lagrangians. The presence of the generalized kernel in the equation of motion modulates the time dependence of the friction coefficient and of the spring equation, also in case of Lagrangians with constant parameters  $r$  and  $s$ . In this case, the Hamiltonian has a sign depending on the momentum  $p$  and the coordinate  $x$  in a very simple way. This “comfortable” Hamiltonian and the role of the kernel in the Hamilton equations, can be useful to understand the complex dynamics of dissipative systems. The Euler–Lagrange equations show a perfect equivalence with the corresponding Hamilton equations, supporting the procedure suggested in this work. Moreover, the memory effect in the equation generated by the kernel can avoid insertion of *ad hoc* time dependent coefficients in the Lagrangian, in order to explain different kinds of time-dependent dissipative systems.

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